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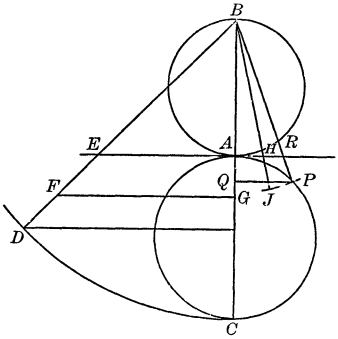
2734 [1918, 444]. Proposed by E. L. REES, University of Kentucky.

Given two circles tangent to each other externally. From the extremity of a diameter through the point of tangency draw a secant such that the segment between the circles shall be equal to a given segment.



SOLUTION BY THE PROPOSER.

Let PB be the required position of the secant, assuming the problem having been solved. Draw PQ perpendicular to AC and let $BP = y$, $AQ = x$, $AB = d$, $AC = d'$, and $RP = a$, the given segment. Then we have, $y^2 = (d + x)^2 + x(d' - x)$. Also $(y - a)/d = (d + x)/y$. Eliminating x, we get $y^2 - 2a'y - d^2 = 0$, where $a' = a(2d + d')/2(d + d')$. Hence,

$$y = a' \pm \sqrt{a'^2 + d^2}.$$

The analysis suggests the following construction: With B as a center, BC as a radius, describe the arc cutting the horizontal diameter produced of the lower circle in D. Draw BD cutting the common tangent in E. Take $EF = a$. Draw FG perpendicular to AC. Then AG will equal a'. Make $AH = AG$. Draw BH and produce it to J, making $HJ = AH$. With B as a center and BJ as radius, describe an arc cutting the lower circle in P. Then BP is the required secant.

Also solved by P. J. DA CUNHA.

2741 [1919, 35]. Proposed by H. L. OLSON, New Hampshire College.

Prove or disprove the following statement: If the three sides and the area of a triangle are integers, at least one of the three altitudes is an integer.

I. SOLUTION BY FRANK IRWIN, University of California.

The statement is not true since the triangle whose sides are 5, 29, 30 has area 72, and altitudes 144/5, 144/29, and 24/5.

II. REMARKS BY J. L. RILEY, Stephenville, Texas.

On page 12 of Carmichael's *Diophantine Analysis* we find the following theorem:

A necessary and sufficient condition that rational numbers x, y, z shall represent the sides of a rational triangle is that they shall be proportional to numbers of the form $n(m^2 + h^2)$, $m(n^2 + h^2)$, $(m + n)(mn - h^2)$, where m, n, h are positive rational numbers and $mn > h^2$.

In deriving this result it is pointed out that if $x = n(m^2 + h^2)$, $y = m(n^2 + h^2)$, and $z = (m + n)(mn - h^2)$, the area is $hmn(m + n)(mn - h^2)$ and the altitude upon z is $2hmn$. If m, n, and h are integers ($mn > h^2$) we have a series of triangles for which the statement of Mr. Olson is true.

2742 [1919, 36]. Proposed by C. N. SCHMALL, New York City.

In Gregory's *Examples in the Differential and Integral Calculus*, 1841, Chap. VII, p. 124, ex. 22, I find the following celebrated problem: "To find a point within a triangle from which if lines be drawn to the angular points their sum may be the least possible." The author remarks that "the direct solution of this problem is long and complicated, etc." Required a simple, brief solution.

I. SOLUTION BY F. V. MORLEY, Johns Hopkins University.

Consider the three points, α, β, γ (complex variable) as on a unit or base circle. Then th sum of the distances from the point x is

$$\sum^{3} \sqrt{(x - \alpha)\left(\bar{x} - \frac{1}{\alpha}\right)} \tag{1}$$

where $x - \alpha = \rho t$, the distance times the turn, or direction factor, and $\bar{x} - 1/\alpha = \rho/t$, the conjugate expression. Since (1) is to be a minimum, D_x and $D_{\bar{x}}$ must vanish. As they are interdependent, either will suffice.

$$D_{\bar{x}} = \sum^{3} \sqrt{\frac{x - \alpha}{\bar{x} - 1/\alpha}} = \sum^{3} t = 0. \tag{2}$$

This means that sum of the three roots (the symmetric function s_1) vanishes in the equation

$$t^3 - s_1 t^2 + s_2 t - s_3 = 0. \tag{3}$$

The condition $D_x = 0$ involves $s_2 = 0$, so that equation (3) reduces to

$$t^3 - s_3 = 0$$

the roots of which are

$$t, \ \omega t, \ \omega^2 t,$$

when ω is a cube root of 1. That is, the three angles at x are all equal to 120°.

The argument may also be presented in its physical, instead of geometrical, aspect. The method in each case is the same.

Consider α, β, γ as centers of attraction for constant forces of equal magnitude. The total potential of x will then be the sum of the distances from x to the three points. For the total potential to be a minimum is to have equilibrium, and the resultant force in any direction must vanish. Hence, the three angles at x are all equal to 120°. It is to be noted that the partial derivative ($D_{\bar{x}}$) of potential, expresses the resultant force with direction factor included.

The well known construction for (the Fermat point) x is to draw on each side $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$, an equilateral triangle, outward, calling the free vertices γ', α', β'. The points γ', α, x, β, are concyclic, and it follows that the line $\gamma' x$ bisects the angle $\alpha x \beta$ and coincides with the line $x\gamma$. x will therefore be the intersection of the three lines $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$.

II. Solution by R. E. Moritz, University of Washington.

Let $P_i = (x_i, y_i)$, $i = 1, 2, 3$, represent the angular points of the triangle expressed in Cartesian coördinates, $P = (x, y)$ any point within the triangle, and θ_i the angle which the line P_iP makes with the positive direction of the x-axis. Then, if D denotes the sum of the distances of P from the angular points, we have

$$D = \Sigma_i \sqrt{(x - x_i)^2 + (y - y_i)^2}.$$

In order that D may be a minimum the partial derivatives of D with respect to x and y must be separately equal to zero; hence,

$$\Sigma_i \frac{x - x_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} = 0, \qquad \Sigma_i \frac{y - y_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} = 0,$$

that is,

$$\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0, \qquad \sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0,$$

or

$$\cos\theta_1 + \cos\theta_2 = -\cos\theta_3, \qquad \sin\theta_1 + \sin\theta_2 = -\sin\theta_3.$$

By squaring and then adding the resulting equations we get $2\cos(\theta_1 - \theta_2) = -1$, from which $\theta_2 - \theta_1 = 120°$. Similarly, $\theta_3 - \theta_2 = 120°$, $\theta_1 - \theta_3 = 120°$.

The problem, therefore, resolves itself into that of finding a point within the triangle at which the three sides subtend equal angles. This point is readily found as follows:

Describe on two sides of the triangle segments of circles containing each an angle of 120°. The intersection point of these circles is the required point. If one of the angles of the triangle is equal to or greater than 120° the intersection point falls on or without the triangle. In that case no point within the triangle satisfies the required conditions.

The following kinematic solution, while perhaps not simpler than the foregoing geometrical solution, has the advantage of being applicable to the more general problem, which I believe has never been solved, namely,

Given n points in a plane, to find a point from which if lines be drawn to the n points their sum may be the least possible.

Draw the triangle to a given scale on a drawing board and insert thumb tacks at the points marking the vertices. Take a flexible string and fasten one end of it to one of the thumb tacks, say to A. Pass the other end of the string through a ring R, then around the thumb tack B, then back through the ring, then around the thumb tack at C, then once more through the ring R and then back to A. Now pull the free end of the string taut until the ring assumes a fixed position. The center of the ring R will be the required point. The proof is obvious.

III. Note by William Hoover, Columbus, Ohio.

The following extract from a paper by P. G. Tait[1] meets the requirement fully and is of interest historically:

The following problem, originally proposed by Fermat to Torricelli, *To find the point the sum of whose distances from three given points is the least possible*, seems to have given considerable trouble to the older mathematicians, and even in modern times (see *Gregory's Examples*, p. 126) to have been solved in a very tedious manner. Simpler solutions have since been given (*e.g.*, *Cambridge and Dublin Mathematical Journal*, VIII, p. 92), but none, to my knowledge, so direct, as that indicated by Quaternions. The object of this note is to show the simplicity of the quaternion method.

If α, β be the vectors of two of the given points, the origin being the third, and if ρ be the vector of the required point, we must have (by the conditions of the problem)

$$T\rho + T(\alpha - \rho) + T(\beta - \rho) \text{ a minimum.}$$

Hence,

$$S[U\rho - U(\alpha - \rho) - U(\beta - \rho)]d\rho = 0,$$

for all values of $Ud\rho$. Hence the versor sum in square brackets must vanish identically. The immediate interpretation is, that *lines parallel* to ρ, $\rho - \alpha$, $\rho - \beta$ form an equilateral triangle. The required point is therefore in the same plane as the three given points; and their distances, two and two, subtend equal angles at it, which is the well-known solution.

Equally simple is the quaternion solution of the same problem if more than three points be given. Let their vectors, to any origin, be α, β, γ, etc., and let ρ be the vector of the sought point. We have

$$\Sigma \cdot T(\alpha - \rho) = \text{minimum},$$

from which, as above, $\Sigma U(\alpha - \rho) = 0$.

Hence, *if unit forces act at the required point, in the lines joining it with the given points, these forces are in equilibrium.* Or, in another form, *a closed equilateral gauche polygon may be drawn whose sides are parallel to the lines joining the sought point with the given ones.*"

IV. Solution by Otto Dunkel, Washington University.

Suppose that at P, a point inside the triangle ABC, we have a minimum with $AP = r_1$, $BP = r_2$, $CP = r_3$. Then r_1 must be the minimum distance from A to the ellipse with foci B and C and passing through P, since for any point Q on the ellipse $BQ + CQ = r_2 + r_3$. But the shortest distance from an external point to a closed convex curve is orthogonal to the curve and we know from the properties of the ellipse that r_1 in this case must make equal angles with the focal rays r_2 and r_3. Applying the same argument to r_2, we see that r_1, r_2, r_3 must make equal angles with one another.

V. Remarks and Historical Notes by R. C. Archibald, Brown University.

This problem has been already discussed in the Monthly by Professors Jackson[2] and Johnson.[3] Professor Jackson's discussion is about the same as Gregory's. The analytic treatment by Professor Moritz is practically identical with that given in 1853 by A. Cohen[4] and in 1902 by E.

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[1] *Proceedings of the Royal Society of Edinburgh*, Vol. 6 (1866–69), 1869, pp. 165–166; also in P. G. Tait, *Scientific Papers*, Vol. 1, 1898, pp. 76, 77.

[2] Volume 24 (1917), pp. 42–44.

[3] Volume 24 (1917), pp. 243–244.

[4] *Cambridge and Dublin Mathematical Journal*, Vol. 9, p. 92.

Goursat,[1] and his kinematic solution is given in the latter part of Tait's paper of 1867 from which Professor Hoover quoted. Tait concludes: "This kinematical process, equally with the quaternion one whose form directly suggests it, gives easily the solution of the more general problem,—To find a point such that m times its distance from A, together with n times its distance from B, etc., may be a minimum." (For three points this is solved in Stegmann–Kiepert, *Lehrbuch der Differentialrechnung*, 7. Aufl. 1895, p. 260.) The problem of finding a point in a plane the sum of whose distances from any number of given points is a minimum was solved by Tédenat in *Annales de mathématiques pures et appliquées* (1810–11), pp. 285–291.

The celebrated problem of our question was formulated by Fermat[2] in the seventeenth century. According to Viviani[3] he suggested it to Torricelli (before 1648, for Torricelli died in 1647). Torricelli discovered three solutions, one by "plane loci" (that is, with ruler and compasses), two others by "solid loci" (that is, by means of conic sections). He afterwards proposed it to Viviani in the following terms: "A triangle, each of whose angles is less than one-third of four right angles[4] being given; to find a point from which, if straight lines be drawn to its three angles, their sum shall be a minimum."

Viviani solved the problem after repeated efforts (he says, non nisi iteratis oppugnationibus tunc nobis vincere datum fuit) and his solution is given in the appendix to his *Geometria divinatio*.[5]

The problem was treated by Thomas Simpson[6] in 1750. He gave the following construction for determining the Fermat point: Describe on BC a segment of a circle to contain an angle of 120°, and let the whole circle BCQ be completed. From A to Q_1 the middle point of the arc $BA'C$ draw AQ intersecting the circumference of the circle in P, which will be the point required. Because of this construction Simson has been credited with the theorem: If on the sides of a triangle ABC, equilateral triangles $A'BC$, $B'CA$ and $C'AB$ be described externally AA', BB' and CC' are concurrent[7]—a construction which Mr. Morley cites above.

Simpson treats also the more general problem: Three points A, B, and C being given, to find the position of a fourth point V, so that if lines be drawn from thence to the three former, the sum $a \cdot AP + b \cdot BP + c \cdot CP$, where a, b, c denote given numbers, shall be a minimum.

The further history of generalizations is very extensive.

Also solved by P. J. DANIEL, W. W. GORSLINE, R. A. JOHNSON, H. M. ROESER, J. ROSENBAUM, L. WEISNER and the PROPOSER.

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[1] *Cours d'analyse mathématique* tome 1; English ed. by Hedrick, 1904, pp. 130–131. See also G. Humbert, *Cours d'Analyse*, tome 1, 1903, pp. 193–196.

[2] *Oeuvres*, Tome 1, Paris, 1891, p. 153: "Datis tribus punctis, quartum reperire, a quo si ducantur tres rectæ addata puncta, summa trium harum rectarum sit minima quantitas." See also Tome 3, 1896, p. 136.

[3] V. Viviani, *De maximis et minimis geometria divinatio*. Florentiæ, MDCLIX, p. 144.

[4] This careful statement is necessary for constructions indicated in solutions above. If one of the angles, A say, is equal to, or greater than, 120°, in a certain sense its vertex is to be considered as the minimum point; compare the discussion of this by Goursat, *l.c.*, by Bertrand in *Journal de mathématiques pure et appliquées*, tome 7 (1843), pp. 155–160, and by Sturm in *Journal für die reine und angewandte Mathematik*, Vol. 97 (1884), p. 51.

[5] *L.c.*, pp. 145–150. Viviani's method of solution is reproduced in D. Cresswell's *Maxima and Minima*, Cambridge, 1812, pp. 120–121; second edition, 1817, pp. 121–122.

[6] *Doctrine and Applications of Fluxions*, London, 1750, § 36.

[7] The first formulation of the result as here stated seems to have been by T. S. Davies in the *Gentleman's Diary* for 1830, p. 36. It is here shown also that $AA' = BB' = CC'$. Problems closely related to this are frequently published, *e.g.*, in *School Science and Mathematics*, Feb., 1918, pp. 170–171; Jan., 1919, pp. 86–87; April, 1919, pp. 374–375.